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1985 J. Phys. A: Math. Gen. 18 L1087

(http://iopscience.iop.org/0305-4470/18/17/005)

J. Phys. A: Math. Gen. 18 (1985) L1087-L1093. Printed in Great Britain

LETTER TO THE EDITOR

Exact results for a checkerboard Ising model with crossing and four-spin interactions

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Received 13 September 1985

Abstract. A seven-parameter Ising model with crossing and four-spin interactions on a checkerboard lattice is mapped by a duality transformation onto a general eight-vertex model, which has only five independent parameters. In a six-dimensional subspace of the Ising model (for which the corresponding eight-vertex model satisfies the free fermion condition) an exact solution is found.

Spin and vertex models formulated on a checkerboard lattice are more general and difficult to treat than those defined on a square lattice. This is caused by the staggered structure of the checkerboard lattice. The only exact solutions of systems on this type of lattices are the Ising model with nearest-neighbour (NN) interactions (Utiyama 1951) and the critical q state Potts model with NN interactions (Maillard and Rammal 1983). Moreover while an exhaustive classification of the solutions of the star-triangle relation for two-component vertex and spin models on the square lattice has been carried out (Krichever 1981, Sogo *et al* 1982, Maillard and Garel 1984), such a classification has not yet been done for staggered models. (As is well known, if the statistical weighs of a two-dimensional model satisfy the star-triangle relation, the model can be exactly solved (Baxter 1982).) Hence it is of interest to obtain exact results for this type of staggered model. In this letter an exact solution of an Ising model with crossing and four-spin interactions, formulated on a checkerboard lattice, is presented. The model is defined as follows. Consider $2N$ Ising spins on a square lattice with periodic boundary conditions, with a Hamiltonian having a checkerboard-type symmetry, as shown in figure 1.

The four spins σ, σ', σ'' and σ''' surrounding each shaded square of figure 1 interact with spin-reversal invariant interactions, which can be written, in the most general case, as

$$E(\sigma, \sigma', \sigma'', \sigma''') = -J_1\sigma\sigma'' - J_2\sigma'\sigma''' - J_3\sigma''\sigma''' - J_4\sigma\sigma' - J_5\sigma\sigma''' - J_6\sigma'\sigma'' - J_7\sigma\sigma'\sigma''\sigma'''. \tag{1}$$

The interactions are depicted in figure 2. The partition function is

$$Z = \sum_{\sigma} \exp\left(\sum{}' - \beta E(\sigma, \sigma', \sigma'', \sigma''')\right) \tag{2}$$

where Σ' indicates a sum over all the shaded squares of the lattice, and $\beta = 1/k_B T$.

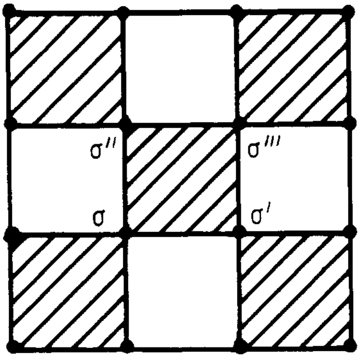

Figure 1. General checkerboard lattice. Each shaded square contains the interactions shown in figure 2.

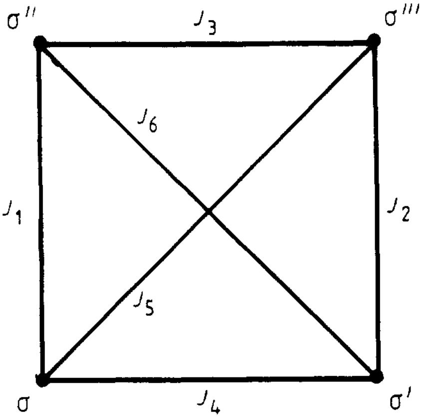

Figure 2. Interactions contained in a shaded square in figure 1 (the four-spin interaction J_7 is not shown).

The geometrical invariances of the lattice under 90° and 180° rotations, as well as specular reflection, lead to the following symmetry relations for the partition function

$$\begin{aligned} Z_{123456} &= Z_{431265} \qquad & Z_{1234} &= Z_{2143} \\ Z_{1256} &= Z_{2165} \qquad & Z_{3456} &= Z_{4356} \end{aligned} \tag{3}$$

where each subscript $i(=1, \ldots, 6)$ stands for the dependence on the parameters K_i.

Moreover, taking into account the periodic boundary conditions, it is evident that the following relations hold

$$\begin{aligned} Z(K_5+\tfrac{1}{2}\mathrm{i}\pi) &= \mathrm{i}^N Z(K_5) \qquad & Z(K_6+\tfrac{1}{2}\mathrm{i}\pi) &= \mathrm{i}^N Z(K_6) \\ Z(K_7+\tfrac{1}{2}\mathrm{i}\pi) &= \mathrm{i}^N Z(K_7) \qquad & Z(K_j+\tfrac{1}{2}\mathrm{i}\pi, K_l+\tfrac{1}{2}\mathrm{i}\pi) &= (-1)^N Z(K_j, K_l) \end{aligned} \tag{4}$$

where in the last relation $j, l = 1, \ldots, 4$ and $l > j$. The key step of this work is to transform the model (2) into a general eight-vertex model without a staggered structure. With this aim we first relabel the spins σ, as well as geometrically changing their locations on the lattice, which is divided into two sublattices A and B, as shown in figure 3. We denote by σ_1 and σ_2 the spins on sublattices A and B, respectively. Now

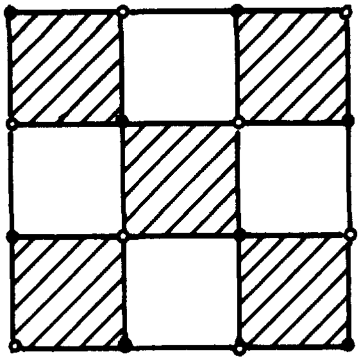

Figure 3. Vertices denoted by full (open) circles form sublattices A (B) respectively.

all the σ_2 spins are translated to their NN site in the right horizontal direction. Thus, in each vertex of sublattice A there are now two spins σ_1 and σ_2. The resulting lattice is sublattice A, i.e. a square lattice, whose elementary cell is shown in figure 4.

The interactions present in the elementary cell are

$$E' = -J_1(\sigma_1\sigma_2'' + \sigma_1'\sigma_2''') - J_2(\sigma_2\sigma_1' + \sigma_2''\sigma_1''') - J_3\sigma_1'\sigma_2'' - J_4(\sigma_1\sigma_2 + \sigma_1'\sigma_2' + \sigma_1''\sigma_2'' + \sigma_1'''\sigma_2''')$$
$$- J_5(\sigma_1\sigma_1' + \sigma_1''\sigma_1''') - J_6(\sigma_2\sigma_2'' + \sigma_2'\sigma_2''') - J_7\sigma_1\sigma_2\sigma_1'\sigma_2'' \tag{5}$$

and the partition function is now given by

$$Z = \sum_{\sigma_1,\sigma_2} \exp\Big(\sum_x K_1\sigma_1\sigma_2'' + K_2\sigma_1'\sigma_2 + K_3\sigma_1'\sigma_2'' + K_4\sigma_1\sigma_2$$
$$+ K_5\sigma_1\sigma_1' + K_6\sigma_2\sigma_2'' + K_7\sigma_1\sigma_1'\sigma_2\sigma_2''\Big) \tag{6}$$

where Σ_x indicates a sum over the sites of the new lattice. Expressed in this form the model does not have a staggered structure, and this fact simplifies the following calculations.

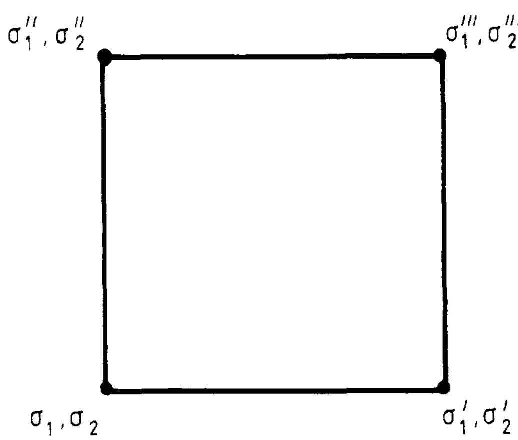

Figure 4. Elementary cell of the resulting lattice after translating the spins σ_2: there are two spins on each vertex.

We now perform a duality transformation on (6) by using a procedure previously introduced (Giacomini 1985). The method is straightforward and only the fundamental steps of the calculations will be given. First the Boltzmann factor in (6) is linearised in the spins σ_1 and σ_2. Then seven new variables, which take the values zero and one, are introduced in order to expand the resulting products. In this way the spins σ_1 and σ_2 can be decoupled and thus exactly decimated. (For further details it is remarked that this type of transformation is extensively discussed in the examples analysed in the papers cited above.)

After expressing the new variables in term of Ising spins, the partition function is given by

$$Z = 2^{-5N} \exp[N(K_1 + \ldots + K_7)] \sum_{\sigma_1,\ldots,\sigma_7} \prod_x (1+\alpha_1\sigma_1)(1+\alpha_2\sigma_2)\ldots(1+\alpha_7\sigma_7) \tag{7}$$

where Π_x indicates a product on the sites of the lattice, $\alpha_i = \exp(-2K_i)$ and the σ_i are Ising spins constrained by the equations

$$\sigma_1\sigma_2'\sigma_3'\sigma_4\sigma_5\sigma_5'\sigma_7\sigma_7' = 1 \qquad \sigma_1''\sigma_2\sigma_3''\sigma_4\sigma_6\sigma_6''\sigma_7\sigma_7'' = 1 \tag{8a, b}$$

which are imposed on each site of the lattice. From (8a) we get

$$\sigma_4 = \sigma_1\sigma_2'\sigma_3'\sigma_5\sigma_5'\sigma_7\sigma_7' \tag{9}$$

and therefore (8b) becomes

$$\sigma_1\sigma_1''\sigma_2\sigma_2'\sigma_3'\sigma_3''\sigma_5\sigma_5'\sigma_6\sigma_6''\sigma_7'\sigma_7'' = 1. \tag{10}$$

This constraint can be simplified by changing the spin variables as follows

$$\sigma_1 \to \sigma_1\sigma_3\sigma_6\sigma_7 \qquad \sigma_2 \to \sigma_2\sigma_3\sigma_5\sigma_7. \tag{11}$$

Taking (9) and (11) into account, the partition function (7) and the constraint (10) are now given by

$$Z = 2^{-5N} \exp[N(K_1 + \ldots + K_7)] \sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_5,\sigma_6,\sigma_7} \prod_x (1+\alpha_1\sigma_1\sigma_3\sigma_6\sigma_7)(1+\alpha_2\sigma_2\sigma_3\sigma_5\sigma_7)$$

$$\times (1+\alpha_3\sigma_3)(1+\alpha_4\sigma_1\sigma_2'\sigma_3\sigma_5)(1+\alpha_5\sigma_5)(1+\alpha_6\sigma_6)(1+\alpha_7\sigma_7) \tag{12}$$

and

$$\sigma_1\sigma_1''\sigma_2\sigma_2' = 1 \tag{13}$$

Now the spins σ_3, σ_5, σ_6 and σ_7 are decoupled in (12) and can be decimated. Moreover the solution of (13) is $\sigma_1 = \sigma\sigma'$ and $\sigma_2 = \sigma\sigma''$ where σ is an Ising spin. Hence, after some lengthy, but straightforward, algebra, (12) is expressed as

$$Z = [\tfrac{1}{2}\exp(K_1 + \ldots + K_7)]^N \sum_\sigma \prod_f w(\sigma, \sigma', \sigma'', \sigma''') \tag{14}$$

where the product is over all faces of the square lattice A, and

$$w(\sigma, \sigma', \sigma'', \sigma''') = 1 + \alpha_2\alpha_3\alpha_5\alpha_7\sigma\sigma'' + \alpha_1\alpha_3\alpha_6\alpha_7\sigma\sigma' + \alpha_1\alpha_2\alpha_5\alpha_6\sigma'\sigma'' + \alpha_1\alpha_4\alpha_5\alpha_7\sigma'\sigma'''$$

$$+ \alpha_3\alpha_4\alpha_5\alpha_6\sigma\sigma''' + \alpha_2\alpha_4\alpha_6\alpha_7\sigma''\sigma''' + \alpha_1\alpha_2\alpha_3\alpha_4\sigma\sigma'\sigma''\sigma''' \tag{15}$$

with σ, σ', σ'' and σ''' being flur spins round a face of the square lattice A, as shown in figure 5.

However $\Sigma_\sigma \Pi_f\, w(\sigma, \sigma', \sigma'', \sigma''')$ is the partition function of the general eight-vertex model expressed in tems of Ising spins (Baxter 1981), with the vertex weights ω_i given

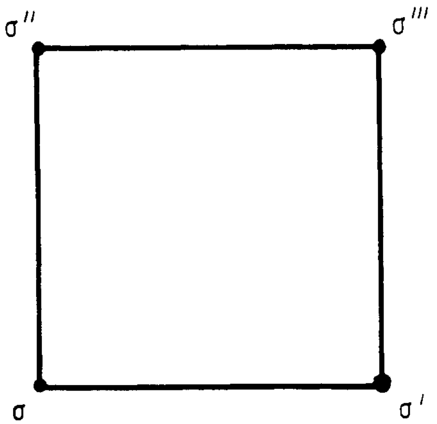

Figure 5. σ, σ', σ'' and σ''' are four spins round a face of the square lattice A. These spins are the final variables in the duality transformation procedure.

by

$$\begin{aligned} &\omega_1 = w(++++) && \omega_5 = w(+++-) \\ &\omega_2 = w(+-+-) && \omega_6 = w(+-++) \\ &\omega_3 = w(+--+) && \omega_7 = w(++-+) \\ &\omega_4 = w(++--) && \omega_8 = w(-+++). \end{aligned} \tag{16}$$

Therefore (14) can be written as

$$Z(K_1, \ldots, K_7) = R^N Z_{8v}(\omega_1, \ldots, \omega_8) \tag{17}$$

where $R = \frac{1}{2}\exp(K_1 + \ldots + K_7)$.

The general eight-vertex model has only five independent parameters, because without loss of generality we can get $\omega_5 = \omega_6$ and $\omega_7 = \omega_8$, and the weights can be multiplied by an arbitrary overall factor (Fan and Wu 1970). Therefore, even though we started with a model with seven independent parameters, the mapping by duality produced a model with only five independent ones. This fact reflects some 'hidden' symmetry of the model, which is not simply related to local properties of the lattice, as is the more common situation in statistical mechanics. This type of 'hidden' symmetry occurs also in the free fermion model (Bazhanov and Stroganov 1985) which is formulated with four independent parameters and the exact solution can be expressed only in terms of three effective variables.

Another example of a 'hidden' symmetry is the S_4 invariance of the q-state checkerboard lattice Potts model (Maillard and Rammal 1984, 1985). In these two cases the symmetry can only be seen from closed expressions for the partition function, while for the model (2) presented in this work, the 'hidden' symmetry can be detected from a duality transformation. On the other hand, the symmetry relations (3) and (4) lead, by using (17), to known symmetry relations of the eight-vertex model (Fan and Wu 1970)

$$\begin{aligned} &Z_{8v} \equiv Z_{8v}(1,2,3,4) = Z_{8v}(3,4,1,2) = Z_{8v}(2,1,4,3) = Z_{8v}(4,3,2,1) \\ &Z_{8v}(1,2,3,4) = Z_{8v}(1,2,4,3) = Z_{8v}(3,4,2,1) = Z_{8v}(2,1,3,4) \\ &Z_{8v}(5,6,7,8) = Z_{8v}(7,8,5,6) \end{aligned} \tag{18}$$

where $i = 1, \ldots, 8$ stands for the dependence on the vertex weights ω_i. There are several interesting particular cases of the duality relation (17). When $K_5 = K_6 = K_7 = 0$ one of the spin variables can be decimated in (6). The anisotropic triangular Ising model results after the decimation. In this case, the corresponding eight-vertex model also becomes the triangular Ising model and the known self-duality relation for this system is obtained. When some of the parameters K_1, K_2, K_3 or K_4 are equal to ∞, the same result is reached. If we take $K_7 = \infty$ in (6) we get the Baxter model exressed in terms of Ising spins (Baxter 1982). In this case the corresponding eight-vertex model obtained by duality also reduces to the Baxter model and the known self-duality relation for this system is obtained. On the other hand, the general eight-vertex model can be exactly solved when the free fermion condition $\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8$ holds (Fan and Wu 1970).

Taking into account (15) and (16), the free fermion condition is satisfied if we impose over the K_i parameters the following constraint

$$\exp[-4(K_5 + K_7)] + \exp[-4(K_6 + K_7)] - \exp[-4(K_5 + K_6)] = 1. \qquad (19)$$

This condition is invariant under the symmetry relations (3) and (4). Therefore we have the exact solution of (2) when the constraint (19) holds. Using the exact expression of the free fermion model free energy (Fan and Wu 1970) and equation (17), the free energy of the model (2) is given by

$$\begin{aligned}\beta f(K_1, \ldots, K_7) &= \lim_{N\to\infty} \frac{1}{N} \log Z \\ &= \log(R) - \frac{1}{8\pi^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \log[a + 2b\cos(\theta) + 2c\cos(\phi) \\ &\quad + 2d\cos(\theta - \phi) + 2e\cos(\theta + \phi)] \qquad (20)\end{aligned}$$

where

$$\begin{aligned} a &= \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 & d &= \omega_3\omega_4 - \omega_7\omega_8 \\ b &= \omega_1\omega_3 - \omega_2\omega_4 & e &= \omega_1\omega_2 - \omega_5\omega_6 \qquad (21) \\ c &= \omega_1\omega_4 - \omega_2\omega_3. \end{aligned}$$

The weights ω_i are given by (16), and the equation (19) must be satisfied. The phase transition condition of the free fermion model is $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 2\max\{\omega_1, \omega_2, \omega_3, \omega_4\}$, and therefore the critical condition for the model (2) is

$$2(1 + \alpha_1\alpha_2\alpha_3\alpha_4) = \max\{\omega_1, \omega_2, \omega_3, \omega_4\} \qquad (22)$$

with ω_1, ω_2, ω_3 and ω_4 given by (15) and (16). This critical variety is invariant under the symmetry relations (3) and (4).

Since the vertex weights are analytic functions of the parameters K_i, the critical behaviour of (2) is the same as the free fermion model. Another soluble case of the eight-vertex model is the Baxter model, which is obtained by setting $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$. However in this case the parameters K_i must satisfy the relations $K_3 = K_4$ and $K_1 = K_2 + \frac{1}{2}\mathrm{i}\pi$ (when $K_7 \neq \infty$; the case $K_7 = \infty$ has been mentioned above), i.e. they take values in a non-physical region. Nevertheless the exact solution of (2) for complex values of the parameters could be of interest in the domain of the mathematical properties of exactly soluble models (Baxter 1981).

Finally, it is of interest to investigate whether the exact solution presented above satisfies the star-triangle relation, as is the case with all known exactly soluble models (Baxter 1982).

I am grateful to Professor Eytan Domany for a critical reading of the manuscript.

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